SOME EXTENSIONS OF WEAKLY MIXING FLOWS

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ABSTRACT

Using a technique of R. Ellis we prove the existence of many weakly mixing (w.m.) flows which are distal extensions of a given w.m. flow. Then we indicate two w.m. minimal flows whose product has a minimal non-w.m. subflow.

0. Introduction

A *flow* (X,ψ) is a compact metric space X with a surjective homeomorphism $\psi: X \to X$. When no confusion occurs we use X rather than (X, ψ). If $Y \subseteq X$ is a ψ-invariant closed set then (Y,ψ) is a *subflow* of (X,ψ). The *orbit* of $x \in X$ is the set $\{\psi^n(x): n = 0, \pm 1, \pm 2, \cdots\}$. X is *minimal* if every orbit is dense and it is *semi-simple* if every orbit closure is a minimal subflow. X is *ergodic* if there exists a dense orbit. Whenever $(X \times X, \psi \times \psi)$ is ergodic X is *weakly mixing* [3].

Let π be a continuous function from X onto Y. π is a *homomorphism* of the flows (X,ψ) and (Y,ϕ) if $\pi\psi = \phi$. (X,ψ) is called an *extension* of its *factor* (Y,ϕ). Two points $x_1, x_2 \in X$ are *proximal* if the orbit closure of (x_1, x_2) in $(X \times X, \psi \times \psi)$ intersects the diagonal. X is *distal* if every point is proximal only to itself. An extension $\pi: (X,\psi) \to (Y,\phi)$ is a *distal extension* if no two different points of X in the same fiber over Y are proximal. Two minimal flows (X,ψ) and (Y,ϕ) are *disjoint* [3] if their product $(X \times Y, \psi \times \phi)$ is minimal. C, R, and Z denote, respectively, the unit circle, the real numbers, and the integers.

It was proved in [3] that every w.m. minimal flow is disjoint from every distal flow. This shows that the distal flows are extremely different from the w.m. flows. Thus, the question arises whether a distal extension of a w.m. flow can still be

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w.m. Theorem 1 asserts by a category argument that this is in fact usually the case. It is easy to show that the product of two w.m. minimal flows is w.m. In [5] Veech asks whether every minimal subflow of such a product is w.m. Lemma 3 provides, again by a category argument, a counter-example. We do not know whether such a product contains at all minimal w.m. subflows. (A positive answer would imply the existence of a universal w.m. minimal but non-metric flow, i.e., a w.m. minimal flow which has every w.m. minimal flow as its factor). We introduce the weakly mixing functions and find in Lemma 5 some conditions for the product of a w.m. function with an almost periodic function to be w.m. A concrete example for the situations dealt with in Theorem 1, Lemma 3, and Lemma 5 follows Lemma 5. Finally, we bring a category theorem concerning the disjointness of extensions.

1. Extensions

Now we introduce the set-up of [1]. Let G be a complete metric topological group and assume (X, ψ) to be a G-extension of (Y, ϕ) (i.e. G acts on X as a group of homeomorphisms commuting with ψ, $Y = X/G$ and ϕ is induced by ψ) with $\pi : X \to Y$. Provide $C(Y,G)$, the set of continuous functions from Y to G, with the metric d of uniform convergence and call G admissible if the following conditions are fulfilled

i) For every $x \in X$ with a dense orbit and open sets $V_1,\cdots,V_n \subset G$ there exists an integer k such that whenever $\{W_1,\cdots,W_k\} \subseteq \{V_1,\cdots,V_n\}$ then $G_x W_1 \cdots W_k = G$ where $G_x = \{g \mid gx = x\}$.

ii) If $u \in C(Y, G)$ and $\varepsilon > 0$ then there exists $\delta > 0$ so that every function w' from some finite subset $F \subset Y$ to G which satisfies $d(u(y),w'(y)) < \delta$ $(y \in F)$ has an extension $w \in C(Y, G)$ with $d(u,w) < \varepsilon$.

Proposition 2 in [1] asserts that every connected Lie group whose left and right uniform structures coincide is admissible.

Every $u \in C(Y,G)$ induces a homeomorphism $\bar{u}$ on X by: $\bar{u}(x) = u[\phi\pi(x)]\psi(x)$. Lemma 1 of [1] gives the powers of $\bar{u}$

$$\bar{u}^n(x) = u(\phi^n y) \cdots u(\phi y)\psi^n(x) \qquad \text{for } n > 0$$

and $$\bar{u}^n(x) = u(\psi^{n+1} y)^{-1} \cdots u(y)^{-1}\psi^n(x) \qquad \text{for } n < 0$$

where $$y = \pi(x).$$

Thus $(x,\bar{u})$ is an extension of (Y, ϕ) with the homomorphism π.

Let (Y, ϕ) be a w.m. non-trivial flow (not necessarily minimal). The following theorem is a close relative of Lemma 2 in [1]:

THEOREM 1. *If G is admissible then there exists a comeager set $W \subset C(Y,G)$ such that for every $u \in W$ $(X,\bar{u})$ is* w.m.

PROOF. Since Y is w.m. we can find a point $(y_0,y_1) \in Y \times Y$ with a dense orbit. Let x_0,x_1 satisfy $\pi(x_i) = y_i$ $(i = 0,1)$ and let $\mathfrak{U} = \{U_i\}$ be a countable base for $X \times X$. For every $U \in \mathfrak{U}$ define $E(U) = \{u \in C(Y,G)/\exists n : \bar{u}^n(x_0,x_1) \in U\}$. The formulae for $\bar{u}^n$ imply immediately that $E(U)$ is open in $C(Y, G.)$

We proceed to show that $E(U)$ is dense. Let $u \in C(Y, G)$ and $\varepsilon > 0$. Take the suitable δ of condition (ii) and choose an open covering $V_1,\cdots,V_n$of $u(Y) \cup u(Y)^{-1}$ with $\operatorname{diam}(V_i) < \delta$ $(i = 1, \cdots, n)$. By (*i*) we may find an integer k such that $G_{x_i} W_1 \cdots W_k = G$ if $\{W_1, \cdots, W_k\} \subset \{V_1,\cdots,V_n\}$ $(i = 0, 1)$. (X, ψ) is a G-extension of (Y, ϕ) thus π is open and there exists an integer p with $\phi^p(y_0,y_1) \in \pi(U)$. Since Y has no isolated points we may assume $|p| \geqq k$ and for sake of convenience we suppose p is positive. Let $W_j^0,W_j^1 \in \{V_1,\cdots,V_n\}$ satisfy $u(\phi^j y_0) \in W_j^0$, $u(\phi^j y_1) \in W_j^1 (j = 1,\cdots,p)$. Take $g_0,g_1 \in G$ to satisfy $\psi^p(g_0 x_0,g_1 x_1) \in U$. Since $G = G_{x_0} W_1^0 \cdots W_p^0 = G_{x_1} W_1^1 \cdots W_p^1$ g_0 and g_1 have representations: $g_0 = g_0^0 g_1^0 \cdots g_p^0$ $g_1 = g_0^1 g_1^1 \cdots g_p^1$ where $g_0^i \in G_{x_i}$ and $g_j^i \in W_j^i$ $(j = 1,\cdots,p;\ i = 0,1)$. Define w' on $\{\phi^j y_0,\phi^j y_1/j = 1,\cdots,p\}$ by $w'(\phi^j y_i) = g_j^i$. w' is well defined and by (ii) it has an extension w whose distance from u is less than ε. Clearly $w \in E(U)$.

The desired W is obtained now by taking $\cap\{E(U)/U \in \mathfrak{U}\}$.

COROLLARY. *If Y is minimal and G is admissible and abelian then $(X,\bar{u})$ $(\bar{u} \in W)$ is a w.m. minimal G-extension of (Y, ϕ).*

PROOF. $(X, \bar{u})$ is ergodic and semisimple so it has to be minimal.

LEMMA. 2. *Let π and θ be respectively homomorphisms of the minimal flows X and Z onto Y. If X is a distal extension then H:$\{(x,z)/\pi(x) = \theta(z)\}$ is semisimple.*

PROOF. Assume $\pi(x_0) = \theta(z_0)$. It is enough to find a minimal ideal $\tilde{I}$ in $E(H)$ — the enveloping semigroup of H ([2]) and an idempotent $\bar{u} \in \tilde{I}$ which satisfies $\bar{u}(x_0,z_0) = (x_0,z_0)$.

The given commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{\sigma_2} & Z \\ {\scriptstyle\sigma_1}\downarrow & & \downarrow{\scriptstyle\theta} \\ X & \xrightarrow[\pi]{} & Y \end{array}$$

where σ_i are the projections induces a commutative diagram of homomorphisms between the corresponding enveloping semigroups

$$\begin{array}{ccc} E(H) & \xrightarrow{\tilde{\sigma}_2} & E(Z) \\ {\scriptstyle\tilde{\sigma}_1}\downarrow & & \downarrow{\scriptstyle\tilde{\theta}} \\ E(X) & \xrightarrow[\tilde{\pi}]{} & E(Y) \end{array}$$

Let I be a minimal ideal in $E(Z)$ and let $u \in I$ satisfy $u(z_0) = z_0$. Take $\tilde{I}$ to be a minimal ideal in $\tilde{\sigma}_2^{-1}(I)$ and $\tilde{u} \in \tilde{I}$ an idempotent over u ($\tilde{\sigma}_2(\tilde{u}) = u$). Denote $\tilde{u}(x_0, z_0) = (x',z')$. Now, $z' = \sigma_2[\tilde{u}(x_0, z_0)] = \tilde{\sigma}_2(\tilde{u})[\sigma_2(x_0,z_0)] = z_0$ and similarly $x' = \tilde{\sigma}_1(\tilde{u})(x_0)$. Since $\tilde{\sigma}_1(\tilde{u})$ is an idempotent in $E(X)$, x_0 and x' are proximal, but $\pi(x') = \pi[\tilde{\sigma}_1(\tilde{u})(x_0)] = \tilde{\pi}\tilde{\sigma}_1(\tilde{u})[\pi(x_0)] = \tilde{\theta}\tilde{\sigma}_2(\tilde{u})[\pi(x_0)] = \tilde{\theta}(u)[\theta(z_0)] = \theta(z_0) = \pi(x_0)$ and X is a distal extension of Y so that $x_0 = x'$ and $\tilde{u}(x_0,z_0) = (x_0,z_0)$.

Lemma 3 answers a question posed by Veech

LEMMA 3. *There exists two w.m.minimal flows whose product has a minimal non-w.m. subflow.*

PROOF. Take T to be the unit circle and (Y,ϕ) any non-trivial w.m. minimal flow. Define $X = Y \times T$, $\psi(y, t) = (\phi y, t)$ and $s(y, t) = (y, st)$ $(s \in T)$. Thus (X,ψ) is a T-extension of (Y, ϕ). Now let $u \in W \cap (-W)$ where W is the set described in Theorem 1. $x_1 = (X,\bar{u})$ and $X_2 = (X,-\bar{u})$ are w.m. minimal T-extensions of Y (the homomorphisms are the projections π_j).

$X_1 \times X_2$ is the product of two w.m. minimal flows so it is w.m.

Take $\Delta = \{(x_1,x_2)/\pi_1 x_1 = \pi_2 x_2\}$. By Lemma 2 Δ is semisimple but $\tau(\langle y,\gamma\rangle, \langle y,\gamma'\rangle) = \gamma/\gamma'$ shows that (-1) is an eigenvalue for every minimal subflow of Δ so Δ contains no minimal w.m. subflow.

LEMMA 4. *The n-torus T^n supports a w.m. minimal flow* $(2 \leqq n \leqq \aleph_0)$.

PROOF. We proceed by induction:

For $n = 2$ Kolmogorov announced in [4] the existence of a minimal continuous flow (T^2, R) with no continuous eigenfunctions. By [5] this implies w.m. Now one can show that for a suitable α $(T^2, Z\alpha)$ is minimal ($Z\alpha$ is regarded as a subgroup of the group of homeomorphisms R). Since $Z\alpha$ is syndetic in R with its usual topology $P(T^2, Z\alpha) = P(T^2, \mathrm{R})$ and by [5] $(T^2, Z\alpha)$ is w.m.

The proof of the preceding Lemma indicates the induction step and for $\aleph_0$ the flow is obtained by taking an inverse limit.

2. Weakly mixing functions

Call a function f from Z to the unit circle C w.m. if the flow X_f it generates is w.m. (X_f is the weak closure of the translates of f). We denote by C_λ $(\lambda \in C)$ the closed group generated by λ. The next lemma is a partial answer to the question when the product of a w.m. minimal function with an almost periodic function is w.m.

LEMMA 5. *Let f be a w.m. minimal function. If $1 \neq \lambda \in C$ and $g(n) = \lambda^n f(n)$ is w.m. then $C_\lambda f \cap X_f \supsetneqq \{f\}$. When λ is of prime order the converse holds too.*

PROOF. Define $\delta : X_f \times C_\lambda \to X_g$ by $\delta(h,\gamma)(n) = \lambda^n \gamma h(n)$. Since $X_f \times C$ is minimal [3] δ is a homomorphism *of* $X_f \times C_\lambda$ onto X_g. Obviously δ is one-to-one iff $C_\lambda f \cap X_f = \{f\}$.

Now if g is w.m. δ is not 1-1 so $C_\lambda f \cap X_f \supsetneqq \{f\}$.

If λ has a prime order and g is not w.m. then X_g has an equicontinuous factor which must be a factor of C_λ and therefore coincide with it. Assume $\tau : X_g \to C_\lambda$ with $\tau(g) = 1$, and define $\rho : X_g \to X_f$ by $\rho(g')(n) = [\tau(g')\lambda^n]^{-1} g'(n)$. Since X_f is w.m. $X_f \times C_\lambda$ is a factor of X_g and $\delta^{-1} = (\rho, \tau)$ so that δ is one-to-one and $C_\lambda f \cap X_f = \{f\}$.

We are going now to build a w.m. sequence f such that $(-1)^n f(n)$ is w.m.

Let (Y,T) be a w.m. minimal flow. $X = Y \times \{-1,1\}$ and $\mathrm{S}(y,\varepsilon) = (Ty, r(y)\varepsilon\}$ where r is a continuous function from Y to $\{-1,1\}$. Thus (X,S) is a group extension of (Y,T). By [6] (X,S) is minimal iff the equation $g(Ty) = r(y)g(y)$ has no continuous solution $g : Y \to \{-1,1\}$. Assume now that (X,S) is minimal but not w.m. Then (X,S) has an eigenfunction $\psi(Sx) = e^{i\lambda}\psi(x)$. Define $\tau(y) = \psi(y,1) \cdot \psi(y,-1)$, then $\tau(Ty) = e^{2i\lambda}\tau(y)$ and since Y is w.m. $e^{2i\lambda} = 1$ so that $\lambda = \pi$. Now ψ is a non-constant invariant function for (X, S^2) which is a group extension of (Y, T^2) so by [6] the equation $g(T^2 y) = r(y)r(Ty)g(y)$ has a continuous solution $g : Y \to \{-1,1\}$. Define $h = Tg \times g = Tg/g$. Then $Tr/r = Tr \times r = Th/h$ and

$T(r/h)=r/h$. Thus r/h is constant and since r is not of the form Tg/g $h = -r$, we have proved the following

LEMMA 6. *(X,S) is weakly mixing minimal iff the equations $Tg = rg$ and $Tg = -rg$ are both unsolvable.*

Consider now Y to be a subflow of $\{-1,1\}^Z$, r the projection on the first coordinate ($r(y) = y_0$) and let $a = (\cdots a_{-1},a_0,a_1,\cdots) \in Y$. If g is a solution of $g(Ty) = r(y)g(y)$ (here T is the shift) then

$$g(Ta) = a_0 g(a),\cdots,g(T^r a) = a_{r-1} g(T^{r-1}a)$$

thence $g(T^r a) = a_0 \cdots a_{r-1} g(a)$. Since g is continuous with a finite range, g depends only on a finite number of coordinates: $g(y) = g(y_{-N},\cdots,y_N)$. Let B be a block of a whose length exceeds $2N+1$. If B appears in a at k_1 and k_2 ($k_1 < k_2$) then

$$g(T^{k_1+N}a) = a_0 \cdots a_{k_1+N-1} g(a)$$

and
$$g(T^{k_2+N}a) = a_0 \cdots a_{k_2+N-1} g(a).$$

Since $T^{k_1+N}a$ and T^{k_2+N} a coincide in their central $(2N+1)$ block the two expressions are equal and $a_0 \cdots a_{k_1+N-1} = a_0 \cdots a_{k_2+N-1}$ which leads to $a_{k_1} \cdots a_{k_2-1} = 1$. We have shown that if the first equation is solvable then there exists an n such that if B is a block of length $\geqq n$ which appears in a at k_1 and k_2 then $a_{k_1} \cdots a_{k_2-1} = 1$ (Condition 1). Similarly one can show that if the second equation is solvable then $a_{k_1} \cdots a_{k_2-1} = (-1)^{k_2-k_1}$ (Condition 2). We conjecture that none of the last two conditions can hold for a w.m. sequence a but we were successful to prove it only for a spuecial seqence. Recall the w.m. sequence built in [5]. It begins with the blocks $A_0 = 1,-1,1$ $B_0 = 1,1,1$ and is built inductively by

$$A_{k+1} = A_k A_k^{(i_m)} A_k \cdots A_k A_k^{(i_2)} A_k \cdots A_k^{(i_m)} A_k$$

$$B_{k+1} = E_k B_k^{(i_m)} E_k \cdots E_k B_k^{(i_2)} E_k \cdots B_k^{(i_m)} E_k$$

where $E_k = 10\cdots 0$, $1 < i_2 < \cdots < i_m$ are the places where 1 appears in B_k and $A_k^{(i_j)}$ is a cyclic permutation of A_k beginning at the i_jth place. By induction one can easily show that the number of (-1)'s which appears in A_k is odd. Since the block $A_k A_k$ appears in A_{k+2} the sequence a cannot satisfy condition 1. Now for every $k > 0$ the block 11 appears at least once in the same place in A_k and B_k. Thus

$$A_{k+1} = \cdots A_k \cdots, 1, 1, \cdots A_k$$

$$B_{k+1} = \cdots E_k \cdots, 1, 1, \cdots E_k.$$

Now we find in A_{k+2} the blocks $\cdots A_{k+1}1,1, \cdots A_k$ and $A_{k+1} \cdots A_k$. This shows that Condition 2 is not satisfied (recall that A_k is the terminal block of A_{k+1}).

The flow (X, S) is isomorphic in this case to the flow generated by the sequence $b = \cdots a_{-2}a_{-1}, a_{-1}, 1, a_0, a_0a_1 \cdots$. A similar construction with $-a$ instead of a, leads to the w.m. sequence $c = \{(-1)^i b_i)\}$ which generates a w.m. minimal flow (Z, T). Now $\{\langle x, z\rangle \mid \alpha(x) = \beta(z)\}$ is minimal $(X \overset{Y}{+} Z)$ and has -1 as an eigenvalue (α and β are the homomorphisms from X and Z res. to Y). (This provides a direct proof of Lemma 3.) A similar argument to that used in Lemma 3 yields

LEMMA 6. *For every sequence* $\{\lambda_i\} \subset C$ *there exists a w.m. minimal function* f *such that* $g_j(n) = \lambda_j^n f(n)$ *is* w.m. *for every* j.

3. Disjointness.

Recall finally the definition of disjointness [2]. Slight changes in the proof of Theorem 1 provide

THEOREM 7. *If* (Y, ψ) *and* (Z, θ) *are minimal disjoint flows, then* $(X, \bar{u})$ *and* (Z, θ) *are disjoint for every* $\bar{u}$ *in a certain comeager subset of* $C(Y, G)$.

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